

# Fermion Ground State of Three Particles in a Harmonic Potential Well and Its Anyon Interpolation

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We examine the detail of the analytic structure of an exact analytic solution of three anyons, which interpolates to the fermion ground state in a harmonic potential well. The analysis is done on the fundamental domain with appropriate boundary conditions. Some remarks on the hard-core conditions and self-adjointness are made.

## I. INTRODUCTION

The concept of an anyon [1] is based on the homotopy group in two space dimensions and induces the idea of a smooth interpolation between bosonic (symmetric) and spinless fermionic (anti-symmetric) spectra. It is believed that the fractional quantum Hall effect is an example of the physical phenomenon of anyons [2]. To realize this theoretically, one usually introduces a statistical gauge field of the Aharonov-Bohm type to the Schrödinger equation for bosonic (or fermionic) particles. This first quantized scheme may trace back [3] to the equation of motion derived from the second quantized Abelian Chern-Simons gauge theory.

The anyon wavefunctions, however, even for a free system, have been hard to get at. The non-trivial exchange property of particles prohibits constructing many particle states from one-particle product states. Due to the discreteness of the energy of the system and the various ways to solve the system, anyonic spectra in a harmonic potential well are widely studied [4,5]. However, the solutions obtained up-to-now are reduced to a subset of the bosonic or the fermionic full spectra when the statistical parameter,  $\alpha$ , is put to the bosonic ( $\alpha = 0$ ) or the fermionic ( $\alpha = 1$ ) limit. (We restrict  $\alpha$  to be between 0 and 1 without loss of any generality). Therefore, a missing state problem arises. Incidentally, the exact solutions obtained analytically so far have a linear energy dependence on the statistical parameter.

One might suspect that the missing state phenomenon is natural since the bosonic limit is supposed to be a singular limit in the sense of the Pauli exclusion principle [6]. However, numerical calculations [7] for a few-body system have demonstrated that interpolating states exist between the missing solutions and that the energy dependence on the statistical parameter might be non-linear. Perturbative investigations [8] seem to confirm this non-linear behavior of the spectra under situations lacking exact information about the interpolation.

Recently, a new family of exact analytic solutions have been proposed in [9], which supplements the known exact solutions. The solutions exhibit a linear energy dependence on the statistical parameter and do not satisfy the hard-core condition in general. However,

the solutions have not been fully accepted in the community. We therefore, present a detailed analysis of the solution, which interpolates to the ground state of three anyons in a harmonic potential well and which corresponds to a typical missing state.

In Section II, we briefly summarize the method to solve the equation. The explicit form of the solution and its properties are given in Section III. Section IV is the conclusion and discussion.

## II. RADIAL AND HARMONIC WAVEFUNCTION

We summarize how to solve the Schrödinger equation for three anyons using the coordinates in Refs. 9-11. Denoting the coordinates of three particles as complex numbers,  $z_a = x_a + iy_a$  with  $a = 1, 2, 3$ , we have the center of mass (CM) coordinate as  $Z = \frac{(z_1 + z_2 + z_3)}{\sqrt{3}}$  and the relative motions (RM) as  $u = \frac{(z_1 + \eta z_2 + \eta^2 z_3)}{\sqrt{3}}$  and  $v = \frac{(z_1 + \eta^2 z_2 + \eta z_3)}{\sqrt{3}}$  where  $\eta = e^{\frac{i2\pi}{3}}$ . The RM can be conveniently parametrized in terms of the Euler angles ( $\xi$ ,  $\theta$ , and  $\phi$ ) and the scale parameter  $r \geq 0$ . That is,  $u = rw$  and  $v = rz$  where  $w = \sin(\xi/2) e^{i(\theta + \frac{\phi}{2})}$  and  $z = \cos(\xi/2) e^{i(\theta - \frac{\phi}{2})}$ .

Typically the Euler angles are defined on  $S^3$  or on  $SU(2)$ , and the angles have the ranges  $0 \leq \xi < \pi$ ,  $0 \leq \chi < 2\pi$ , and  $-2\pi \leq \psi < 2\pi$ . In our case, due to the exchange symmetry, this domain is reduced to  $\frac{S^3}{Z_2 \times Z_3}$ . Note that any two-particle exchange can be represented in a combination of  $E$  and  $P$ , where  $E$  is the second and third particle-exchange operation,  $(1, 2, 3) \rightarrow (1, 3, 2)$ , and  $P$  is a cyclic operation,  $(1, 2, 3) \rightarrow (2, 3, 1)$ . Referring to the definition of  $u$  and  $v$ , we have  $P : (u, v) \rightarrow (\eta^2 u, \eta v)$  and  $E : (u, v) \rightarrow (v, u)$ . In other words, we may choose a fundamental domain given as  $0 \leq r$ ,  $0 \leq \xi < \frac{\pi}{2}$ ,  $-\frac{\pi}{3} \leq \phi < \frac{\pi}{3}$ , and  $0 \leq \theta < 2\pi$  since under  $P$  and  $E$ , we have  $P : (r, \xi, \phi, \theta) \rightarrow (r, \xi, \phi + \frac{2\pi}{3}, \theta + \pi)$  and  $E : (r, \xi, \phi, \theta) \rightarrow (r, \pi - \xi, -\phi, \theta)$ , respectively.

The anyon wavefunction is defined to have a phase  $e^{i\alpha\pi}$  when any of the two particles are interchanged. This requires the wavefunction have the phase under  $P$  and  $E$  [11]

$$\begin{aligned}
E: \quad & \Psi(r, \pi - \xi, -\phi, \theta) = e^{i\alpha\pi} \Psi(r, \xi, \phi, \theta), \\
P: \quad & \Psi(r, \xi, \phi + \frac{2\pi}{3}, \theta + \pi) = e^{i2\alpha\pi} \Psi(r, \xi, \phi, \theta), \\
EPEP: \quad & \Psi(r, \xi, \phi, \theta + 2\pi) = e^{i6\alpha\pi} \Psi(r, \xi, \phi, \theta).
\end{aligned} \tag{1}$$

The interaction energy depends only on the scale parameter  $r$ ;  $V = V(r) = r^2/2$ . This system possesses simultaneous eigenstates of  $H$ ,  $M$ , and  $L$ , where  $H$  is the Hamiltonian for the RM:

$$H = -\frac{1}{4r^3} \frac{\partial}{\partial r} (r^3 \frac{\partial}{\partial r}) + \frac{1}{4r^2} M + V(r). \tag{2}$$

$M$  is a Laplacian on  $S^3$ :

$$M = -\frac{4}{\sin \xi} \frac{\partial}{\partial \xi} \sin \xi \frac{\partial}{\partial \xi} + \frac{1}{\sin^2(\xi/2)} \left( \frac{1}{2i} \frac{\partial}{\partial \theta} + \frac{1}{i} \frac{\partial}{\partial \phi} \right)^2 + \frac{1}{\cos^2(\xi/2)} \left( \frac{1}{2i} \frac{\partial}{\partial \theta} - \frac{1}{i} \frac{\partial}{\partial \phi} \right)^2. \tag{3}$$

Also  $L$  is a relative angular momentum given by

$$L = u \frac{\partial}{\partial u} + v \frac{\partial}{\partial v} - \bar{u} \frac{\partial}{\partial \bar{u}} - \bar{v} \frac{\partial}{\partial \bar{v}} = \frac{1}{i} \frac{\partial}{\partial \theta}. \tag{4}$$

Simultaneous eigenstates of  $H$ ,  $M$ , and  $L$  are given in a factorized form,  $\Psi_{E,\mu,l}(r, \xi, \phi, \theta) = R_{E,\mu}(r) \Xi_{\mu,l}(\xi, \phi, \theta)$  where  $E$ ,  $\mu(\mu + 2)$ , and  $l$  are eigenvalues of  $H$ ,  $M$ , and  $L$ , respectively.  $M$  is a positive semi-definite operator [9,11], and its eigenvalue is semi-positive definite with  $\mu$  being a non-negative number. For the case of the symmetric (bosonic) or the anti-symmetric (fermionic) representation,  $\mu$  is restricted to a non-negative integer. This is because the angular momentum  $L$  takes on integer values. On the other hand, for the anyon representation,  $\mu$  can be fractional since it interpolates bosonic to fermionic states. However, the generic behavior of  $\mu$  is not yet known in terms of the statistical parameter  $\alpha$ . So far, analytically established harmonics have the linear dependence  $\mu = \pm 3\alpha \bmod \text{integer}$ . The new family of solutions suggest that  $\mu = 3\pm\alpha$ .

$\Xi_{\mu,l}$  is a harmonic on  $S^3$ , and all the statistical information is to be encoded on this harmonic since the exchange property in Eq. (1) is independent of the scale parameter  $r$ . Its two-particle analogue is  $e^{\pm i\alpha\theta_{12}}$  where  $0 \leq \theta_{12} \leq \pi$  is the angle of the relative coordinate.

The radial part of the solution can be trivially attained when the harmonics are given. It is given in terms of generalized Laguerre function, and its energy is given as  $E = \mu + 2$ . In the following, we will concentrate on the analysis of the harmonics only. In fact, the same harmonic can be used for obtaining wavefunctions for other problems when the potential energy is scale dependent only. A trivial example is the free anyon case.

### III. HARMONIC WITH $\mu = 3 - \alpha$

The state with  $(\mu = 3 - \alpha, l = -3 + 3\alpha)$  interpolates between the fermionic ground state and a bosonic excited state of the harmonic oscillator. The corresponding harmonic should contain a homogeneous power of  $z$  and  $w$ 's since it is an eigenstate of  $L$ . The solutions are multi-valued, and some subtle points can arise. To investigate the analytic structure reliably, we work on the fundamental domain as given in Section II. The exchange property in Eq. (1) is to be imposed in the form of boundary conditions [10,11]:

$$\begin{aligned}
\Xi(\xi, \phi, \theta = 2\pi) &= e^{i6\alpha\pi} \Xi(\xi, \phi, \theta = 0), \\
\Xi(\xi, \phi = \frac{\pi}{3}, \theta = \pi) &= e^{i2\alpha\pi} \Xi(\xi, \phi = -\frac{\pi}{3}, \theta = 0), \\
\frac{\partial}{\partial \phi} \ln \Xi(\xi, \phi = \frac{\pi}{3}, \theta = \pi) &= \frac{\partial}{\partial \phi} \ln \Xi(\xi, \phi = -\frac{\pi}{3}, \theta = 0), \\
\Xi(\xi = \frac{\pi}{2}, \phi, \theta) &= e^{-i\alpha\pi} \Xi(\xi = \frac{\pi}{2}, -\phi, \theta), \\
\frac{\partial}{\partial \xi} \ln \Xi(\xi = \frac{\pi}{2}, \phi, \theta) &= -\frac{\partial}{\partial \xi} \ln \Xi(\xi = \frac{\pi}{2}, -\phi, \theta).
\end{aligned} \tag{5}$$

It should be noted that the last two identities hold for  $0 < \phi \leq \pi/3$ . In addition, the harmonic is to be normalized, and any current across the boundary needs to be finite, which requires

$$\Xi(\xi = 0, \phi, \theta), \quad \lim_{\phi \rightarrow 0} \phi \Xi^* \frac{\partial}{\partial \xi} \Xi(\xi = \frac{\pi}{2}, \phi, \theta), \quad \frac{\partial}{\partial \phi} \Xi(\xi = \frac{\pi}{2}, \phi = \pm \frac{\pi}{3}, \theta) \tag{6}$$

be finite.

The harmonic on the fundamental domain is given as [9]

$$\Xi_{3-\alpha, -3+3\alpha}(\xi, \phi, \theta) = (z^3 - w^3)^\alpha \Phi(z, w). \tag{7}$$

$\Phi(z, w)$  consists of three terms:

$$\Phi(z, w) = \Phi_0(z, w) + \Phi_1(z, w) + \Phi_2(z, w) \quad (8)$$

where

$$\begin{aligned} \Phi_0(z, w) &= \frac{\bar{z}^3}{(z\bar{z})^{2\alpha}} \frac{(1 + \bar{x})^3}{[(1 + \eta e^{i\pi p} x)(1 + \eta^2 e^{i\pi p} x)(1 + \bar{x})(1 + \bar{\eta}^3 \bar{x})]^\alpha}, \\ \Phi_1(z, w) &= \frac{\bar{z}^3}{(z\bar{z})^{2\alpha}} \frac{(1 + \bar{\eta} \bar{x})^3}{[(1 + \eta^2 e^{i\pi p} x)(1 + e^{i\pi p} x)(1 + \bar{\eta} \bar{x})^2]^\alpha}, \\ \Phi_2(z, w) &= \frac{\bar{z}^3}{(z\bar{z})^{2\alpha}} \frac{(1 + \bar{\eta}^2 \bar{x})^3}{[(1 + e^{i\pi p} x)(1 + \eta e^{i\pi p} x)(1 + \bar{\eta}^2 \bar{x})(1 + \eta \bar{x})]^\alpha}, \end{aligned} \quad (9)$$

and  $x = w/z$ . The factor  $(z^3 - w^3)^\alpha$  satisfies all the anyonic properties given in Eqs. (5) and (6).  $\Phi(z, w)$ , therefore, should satisfy the bosonic boundary condition, which is given in Eq. (5), when  $\alpha = 0$ .

The first boundary condition in Eq. (5) is trivially satisfied. To check the second and the third boundary conditions in Eq. (5), we note that for  $|x| < 1$

$$\begin{aligned} \Phi_k(\eta z, \eta^2 w) &= \Phi_{k+1}(z, w), \\ \Phi_3(z, w) &= \Phi_0(z, w). \end{aligned} \quad (10)$$

Therefore, one can easily see that at  $|\phi| = \frac{\pi}{3}$  and  $0 \leq \xi < \frac{\pi}{2}$

$$\begin{aligned} \Phi(\xi, \phi = \frac{\pi}{3}, \theta) &= \Phi(\xi, \phi = -\frac{\pi}{3}, \theta), \\ \frac{\partial}{\partial \phi} \ln \Phi(\xi, \phi = \frac{\pi}{3}, \theta) &= \frac{\partial}{\partial \phi} \ln \Phi(\xi, \phi = -\frac{\pi}{3}, \theta). \end{aligned} \quad (11)$$

The last two boundary conditions (when  $\xi = \frac{\pi}{2}$ ) are very delicate to check since there are branch points on this boundary at  $\phi = 0, \pm \frac{\pi}{3}$ . To avoid confusion, we are going to work inside the fundamental domain which has no branch points, and, therefore, we can neglect the multi-valuedness. The value at the boundary can be reached by taking the appropriate limit. One can prove that for  $0 < \tau < \frac{\pi}{3}$  and  $\xi = \frac{\pi}{2}^-$ ,

$$\Phi(\xi = \frac{\pi}{2}^-, \phi = -\tau, \theta) = \Phi(\xi = \frac{\pi}{2}^-, \phi = \tau, \theta). \quad (12)$$

More specifically,

$$\begin{aligned}
\Phi_0(\xi = \frac{\pi^-}{2}, \phi = -\tau, \theta) &= \Phi_0(\xi = \frac{\pi^-}{2}, \phi = \tau, \theta), \\
\Phi_1(\xi = \frac{\pi^-}{2}, \phi = -\tau, \theta) &= \Phi_2(\xi = \frac{\pi^-}{2}, \phi = \tau, \theta), \\
\Phi_2(\xi = \frac{\pi^-}{2}, \phi = -\tau, \theta) &= \Phi_1(\xi = \frac{\pi^-}{2}, \phi = \tau, \theta).
\end{aligned} \tag{13}$$

Therefore, all the boundary conditions in Eq. (5) are satisfied except possibly at the branch points, to which we are turning.

At the coincidence limit  $x = 1$ ,  $\Phi_0$  has a smooth limit,  $\Phi_0(\xi = \frac{\pi}{2}, \phi = 0, \theta) = e^{i3\theta} 2^{3/2} 3^{-1/2}$ . However,  $\Phi_1$  and  $\Phi_2$  have a branch cut at  $(\xi = \frac{\pi}{2}, \phi = \frac{\pi}{3})$  and  $(\xi = \frac{\pi}{2}, \phi = -\frac{\pi}{3})$ , respectively, and their values are not determined unambiguously. If we take the limit from inside the domain, we have

$$\lim_{\epsilon \rightarrow 0} \frac{\Phi_1(\xi = \frac{\pi}{2} - \epsilon, \phi = 0, \theta)}{\Phi_2(\xi = \frac{\pi}{2} - \epsilon, \phi = 0, \theta)} = e^{-i\pi\alpha} \tag{14}$$

where we neglect the subtleties due to the branch cuts. If one starts to count multi-valuedness at the branch cuts, then one can change the ratio in Eq. (14) to

$$\lim_{\epsilon \rightarrow 0} \frac{\Phi_1(\xi = \frac{\pi}{2} - \epsilon, \phi = 0, \theta)}{\Phi_2(\xi = \frac{\pi}{2} - \epsilon, \phi = 0, \theta)} = e^{-i\pi(1+2n)\alpha} \tag{15}$$

where  $n$  is an integer representing a branch cut. Therefore, depending on the choice of the branch cut, the hard-core condition can be met at the coincidence limit for  $\alpha = \frac{q}{p}$  where  $q$  and  $p$  are coprimes and odd integers. This is because in this case one can have  $\Phi_1(\xi = \frac{\pi}{2}, \phi = 0, \theta) + \Phi_2(\xi = \frac{\pi}{2}, \phi = 0, \theta) = 0$ . Otherwise,  $\Phi_1$  and  $\Phi_2$  have infinite values at  $x = 1$ , which is to be canceled by the zero of  $(z^3 - w^3)^\alpha$  to make the harmonic finite. The choice of the branch cut, however, looks artificial since the value at the coincidence point is discontinuous from the ones at the surrounding points.

Instead, we claim that the hard-core condition is not a mandatory one for the anyonic system. The Hamiltonian can be self-adjoint even if we do not impose the hard-core condition. The harmonic given in Eq. (7) satisfies all the necessary boundary conditions for the self-adjointness, which are given in Eqs. (5) and (6). Similar behavior is observed in a

two-particle system [12]. The two particles are allowed to collide when self-adjoint extension is done. Here, in the three-particle case, two of the three particles are allowed to collide whereas all the particles cannot collide simultaneously because of the scale-dependent part.

We may construct the harmonic following the series expansion in  $x$  as described in Ref. 11. (Note that  $\nu$  and  $z$  in Ref. 11 correspond to  $\alpha$  and  $x$  in our notation.)  $\mu$  is to be determined by the last two boundary conditions in Eq. (5), which turns out to be the most difficult part in the analysis. We note that this series expansion is equivalent to the approach we use in this paper since both approaches allow the small parameter  $x$ , neglect the multi-valuedness inside the fundamental domain, and have the same boundary conditions to be satisfied. Since there should be a unique solution for the system, the series expansion will reproduce the harmonic in Eq. (7). In this way, we can conclude that the series expansion gives  $\mu = 3 - \alpha$ .

One might suspect that the branch cuts of  $\Phi_1$  and  $\Phi_2$  at the boundary point with  $\xi = \frac{\pi}{2}$  and  $\phi = |\frac{\pi}{3}|$  may give an undesirable multi-valuedness feature. If there is a multi-valuedness contribution, then this should be interpreted as the particle exchange statistical property. However, since that the boundary point corresponds to the particle configuration where the three particles lie on a straight line with particle 2 or particle 3 on the middle of the line, a small circular movement of the middle particle without enclosing any other particle should not give any branch-cut contribution.

Indeed, one can demonstrate that the harmonic has no such branch-cut subtleties. This is because the front factor of the harmonic,  $(z^3 - w^3)^\alpha$  given in Eq. (7), takes care of all the particle-exchange properties, Eq. (1). This gives the true branch cuts on the whole domain. On the other hand, the rest factor,  $\Phi(z, w)$ , is not going to give any anyonic property. Instead, it gives a bosonic property such that the whole expression,  $\Xi$ , has the desired exchange properties. To obey this, we have only to pass on the *value* of  $\Phi(z, w)$  on the fundamental domain to other domains. We recall that  $\Phi(z, w)$  is single-valued on the fundamental domain. Therefore,  $\Phi(z, w)$  on the whole domain never has any multi-valuedness. This is exactly the merit of the harmonic analysis on the fundamental domain.

Of course, the analysis is not complete until one checks the self-adjointness of the system for the analysis to work without any defect. The condition for this is given in terms of the boundary conditions in Eqs. (5) and (6). Since the harmonic satisfies all the necessary boundary conditions, we can conclude that the harmonic has no branch cut at  $(\xi = \frac{\pi}{2}, \phi = |\frac{\pi}{3}|)$  nor any of the subtleties thereof.

#### IV. CONCLUSION AND DISCUSSION

We have presented a detailed analysis for the interpolating solution between the fermion ground state and a bosonic excited state in a harmonic potential well, which is known as a missing state. This solution has a linear energy dependence on the statistical parameter  $\alpha$ .

We claim that the hard-core condition is not mandatory for anyons when two particles coincide with each other. Instead, one can impose the possibility of two particles colliding. Still one can have self-adjointness of the Hamiltonian of the system.

The advantage of the analysis on the fundamental domain is clear. It helps us to avoid unnecessary confusion due to the multi-valuedness of the harmonics. In addition, inside the fundamental domain, one can use a small parameter,  $w/z$  (the ratio of the two relative coordinates), to perform a perturbative analysis for a given statistical parameter.

The tendency to linear behavior of  $\mu$  (the Casimir number of the representation) is expected to continue not only for a three-anyon system but also for a many-anyon system. On the other hand, the issue of smooth interpolation of spectra needs further investigation. This is because the possibility of a many-to-one correspondence of the spectra is not excluded. We are going to report a detailed analysis of this elsewhere.

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